Groups with finitely many conjugacy classes and their automorphisms

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Groups with (nCC) – I

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Cor. *For every $n \in \mathbb{N}$, $n \geq 2$, there exists an infinite countable (nCC)-group G.*

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Remark. If $n \geq 3$ then the examples produced by the above cor. always contain torsion.

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Question 2. Can any countable torsion-free group C, with $x, y \in C \setminus \{1\}$ and $x \overset{C}{\not\sim} y$, be embedded into a t.-f. gp. G with (3CC) s.t. $x \overset{G}{\not\sim} y$?

The answer to Q.2 is "No": set

$$C = \langle a, b \parallel bab^{-1} = a^{-1} \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}.$$

Then $b \overset{C}{\not\sim} b^{-1}$ but if $C \hookrightarrow G$ and $G \models$ (3CC) then $b \overset{G}{\sim} b^{-1}$.

Def. Suppose $x, y \in G$, then x is commensurable with y ($x \overset{G}{\approx} y$) if $\exists\, k, l \neq 0$ s.t. $x^k \overset{G}{\sim} y^l$.

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Setting $C = F(x_1, \dots, x_{n-1})$ we achieve

Cor. *For every* $n \in \mathbb{N}$, $n \geq 2$, *there exists a torsion-free* 2*-generated (*n*CC)-group* G.

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Taking $C = H * F(x_1, \dots, x_{n-1})$ where H is an arbitrary t.-f. gp. one can show:

Cor. *For every $n \in \mathbb{N}$, $n \geq 2$, there exists $2^{\aleph_0}$ of non-isomorphic torsion-free 2-generated 2-boundedly simple (nCC)-groups.*

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Idea of the proof. First take a sequence of HNN-extensions to embed C into a countable t.-f. gp. $\tilde{G}$ that satisfies (ii)-(iv). Then apply small canc. theory over rel. hyp. gps. to embed $\tilde{G}$ into G enjoying (i)-(iv).

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Cor. *The Klein bottle group* $H = \langle a, b \,\|\, bab^{-1} = a^{-1} \rangle$ *can be embedded into a f.g. t.-f. (*4*CC)-gp.* Q *s.t.* $b \stackrel{Q}{\not\sim} b^{-1}$*.*

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(Choose $M = \langle a, b^3 \rangle \lhd H$, $H/M \cong \mathbb{Z}_3$.)

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Cor. $\forall$ *countable gp.* C *there is a f.g. gp.* N *s.t.* N *has (2CC) and Kazhdan's property (T), and* $Out(N) \cong C$.

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Open Problems

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Equivalent Question (B. Neumann). Does there exist a t.-f. (3CC)-gp. G in which no non-trivial element is conjugated to its inverse ?

Question. For $n \geq 2$, does there exist an infinite finitely presented (nCC)-group ?