On Quasiconvex Subsets of Hyperbolic Groups

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The Cayley Graph

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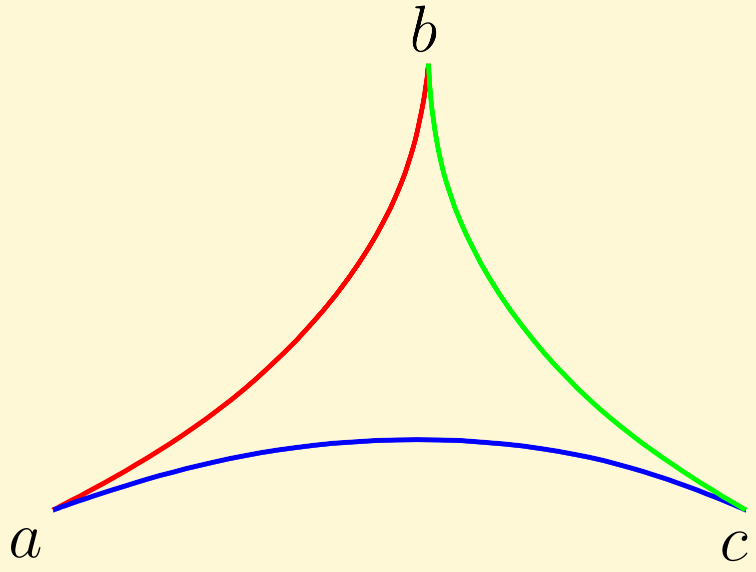
The *Cayley graph* $\Gamma(G, \mathcal{A})$ for the group G with the generating set $\mathcal{A}$ is constructed as follows:

$\Gamma(G, \mathcal{A})$ is a simplicial 1-complex without loops and multiple edges, whose vertices are the elements of G; two vertices x, y are connected by an edge iff $d(x,y) = 1$. Each edge is endowed with the metric of $[0,1]$.

Slim Triangles

A geodesic triangle abc in $\Gamma(G, \mathcal{A})$ is said to be δ-slim if each of its sides belongs to a closed δ-neighborhood of the two others.



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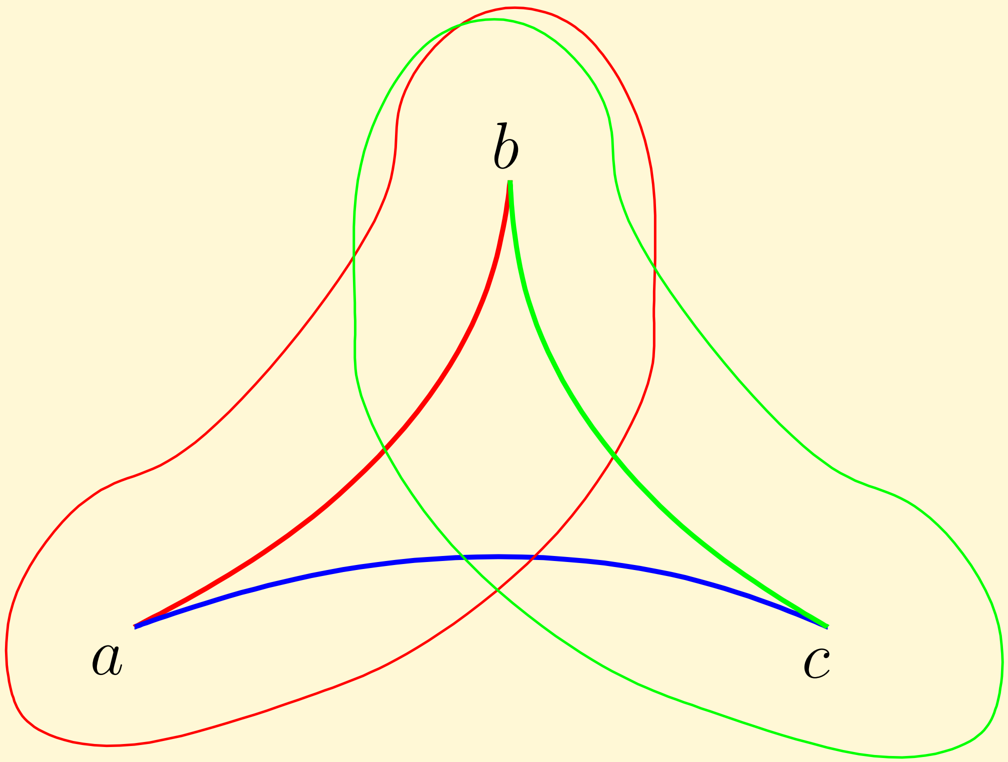

Hyperbolic Groups

Hyperbolic groups were originally introduced by M. Gromov in 1987.

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- all finite groups;
- finitely generated free groups;
- fundamental groups of compact negatively curved Riemannian manifolds;
- f.p. groups satisfying small cancellation conditions (e.g., $C'(1/6)$).

Quasiconvexity: Definition and Examples

Let $\varepsilon \geq 0$ be given.

Def. 2. A subset $Q \subseteq G$ is called *quasiconvex*, if for $\forall\, x, y \in Q$ and any geodesic path p connecting them, $p \subset \mathcal{O}_\varepsilon(Q)$ in $\Gamma(G, \mathcal{A})$.

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5. If $H \leq G$ is infinite and q.c. then $|VN_G(H) : H| < \infty$,

   where $$VN_G(H) = \{g \in G \mid |H : (H \cap gHg^{-1})| < \infty, \\ |gHg^{-1} : (H \cap gHg^{-1})| < \infty\} -$$

   the *virtual normalizer* (*commensurator*) of H in G.

Subset Commensurators

Def. 3. For two subsets A, B of a group G we will write

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Idea of the proof. Use properties of G-action on the boundary ∂G. □

Relatively Small Subsets

Def. 6. A q.c. subset Q of a hyp. group G is *small relatively to* $H \le G$ if there exist no finite $P_1, P_2 \subset G$ such that $H \subseteq P_1 Q^{-1} Q P_2$.

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Theorem 3. $H \leq_{n.e.} G$, $Q \subseteq_{q.c.} G$. *Assume* Q^{-1} *is also q.c. TFAE:*

1. Q *is small rel. to* H;
2. $\forall$ *finite* $P_1, P_2 \subset G$, $H \not\subseteq P_1 Q P_2$.

G-subgroups

Suppose the group G is hyperbolic.

Def. 7. $H \leq_{n.e.} G$ is called a G-*subgroup* if for any finite $M \subset G$ $\exists\, \phi : G \to G_1$, s.t. G_1 is non-elementary hyperbolic, $\phi(H) = G_1$ and $\phi|_M$ is injective.

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$$G = \Big(F(x,y) \times \langle a \rangle_2\Big) * \langle b \rangle,\ H = F(x,y),\ M = \{1, [a,b]\}.$$

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A. Ol'shanskii (1993) found conditions that are both necessary and sufficient for a non-elementary subgroup H to be a G-subgroup.

Homomorphisms Preserving Quasiconvexity

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Idea of the proof. Apply small cancellation theory for hyp. groups developed by Ol'shanskii; use properties of hyp. boundary and rel. small subsets. □

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Theorem 5. *Let G be f.g., $H \leq G$ and $Q \subset G$.*
Suppose Q is not small rel. to H and $\exists\, \phi : G \to G_1$ s.t. $\phi(H) = G_1$ and $\phi|_Q : Q \to \phi(Q)$ is a quasiisometry.
Then H is a quasiretract of G.

Embedding Theorems for Hyperbolic Groups

Theorem 6. [Ol'shanskii, 1995, SQ-universality within the class of all hyperbolic groups]
Let G and H be hyp. groups and G be non-elementary. There exists a non-elementary hyp. quotient G_1 of G containing a copy of H.

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Theorem 9. (Thrifty Embedding) *Suppose G, H are torsion-free hyperbolic groups, G is non-elementary and $H \neq \{1\}$. Then $\exists$ a simple torsion-free quotient M of G and $\pi : H \hookrightarrow M$, s.t. $\pi(H)$ is proper and malnormal in M, and any proper subgroup of M is conjugate (in M) to a subgroup of $\pi(H)$.*