

THE INTEGRAL HOMOLOGY RING OF THE BASED LOOP SPACE ON SOME GENERALISED SYMMETRIC SPACES

JELENA GRBIĆ AND SVJETLANA TERZIĆ

ABSTRACT. In this paper we calculate the integral Pontrjagin homology ring of the based loop space on some generalised symmetric spaces with a toral stationary subgroup. In the Appendix we show that the method can be applied to other type generalised symmetric spaces as well.

1. INTRODUCTION

A pointed topological space X with multiplication $\mu: X \times X \rightarrow X$ is called an H -space. The multiplication induces a ring structure in homology $H_*(X)$. A based loop space ΩX is an H -space as one of the possible multiplications is given by loop concatenation. The ring structure in $H_*(\Omega X)$ induced by loop multiplication is called the Pontrjagin homology ring. In this paper we compute the integral Pontrjagin homology ring of the based loop space on some generalised symmetric spaces G/S , where G is a simple compact Lie group G having the torus S as a stationary subgroup. More precisely, we consider generalised symmetric spaces which have zero Euler characteristic, that is, for which the torus S is not maximal in G . Thus this paper can be considered as a natural sequel to paper [7], where the authors computed the integral Pontrjagin homology of the based loop space of complete flag manifolds of compact simple Lie groups.

Generalised symmetric spaces G/H are defined by the condition that their stationary subgroup H is the fixed point subgroup of a finite order automorphism of the group G . These spaces consequently admit finite order symmetries making them to have rich geometry and thus attract constant attention of many geometry focused research starting with [20],[21] until recent once, see for example [1],[2],[10]. Generalised symmetric spaces play important role in the theory of homogeneous spaces and, thus, in many areas of mathematics and physics such as representation theory, combinatorics, string topology. In particular, complex homogeneous spaces G/H , which are important examples in complex cobordisms and theory of characteristic classes, were characterised by Passier [16] as the homogeneous spaces of positive Euler characteristic whose isotropy subgroup H is a fixed point subgroup for some odd order finite subgroup of the inner automorphisms of the group G and thus are complex generalised symmetric spaces.

The methods used in this paper are analogous to that of [7]. By [10] and [18] all generalised symmetric spaces are Cartan pair homogeneous spaces and, thus, formal in the sense of Sullivan (see Subsections 3.1, 3.2). Therefore, starting from their rational cohomology algebras and using the method of rational homotopy theory together with the Milnor and Moore theorem which expresses the rational Pontrjagin homology algebra in terms of the universal enveloping algebra of the rational homotopy Lie algebra, we compute the rational homology algebras for the based loop space on generalised symmetric spaces G/S .

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We prove that the integral loop homology groups of these spaces have no torsion which together with the rational computations enables us to determine their integral Pontrjagin ring structure as well. In particular, we calculate the integral Pontrjagin homology ring of the based loop spaces on the following generalised symmetric spaces: $SU(2n+1)/T^n$, $SU(2n)/T^n$, $SO(2n+2)/T^n$, $SO(8)/T^2$, and E_6/T^4 .

To illustrate that our approach is more general, in the Appendix we include the analogous calculations for the generalised symmetric space $U(n)/(T^k \times U(n-k))$ which obviously is not obtained by taking a simple compact Lie group and quotienting out a toral subgroup.

2. TORSION IN LOOP SPACE HOMOLOGY

Proposition 2.1. *Let G be a compact connected Lie group and H its closed connected subgroup such that G/H is simply connected. Then $H_*(\Omega(G/H); \mathbb{Z})$ is torsion free if and only if $H_*(\Omega(G/(T^k \times H)); \mathbb{Z})$ is torsion free, whenever $T^k \times H$ is a subgroup of G , where $k \leq \text{rank } G - \text{rank } H$.*

Proof. There is a fibration sequence

$$\Omega(G/H) \longrightarrow \Omega(G/(T^k \times H)) \longrightarrow T^k \xrightarrow{*} G/H \longrightarrow G/(T^k \times H)$$

which implies the homotopy decomposition

$$\Omega(G/(T^k \times H)) \simeq T^k \times \Omega(G/H)$$

and the proof of the proposition. It is important to remark that although the spaces $\Omega(G/(T^k \times H))$ and $T^k \times \Omega(G/H)$ are H -spaces, the decomposition is given only in terms of topological spaces ignoring their H -space structure, that is, this decomposition is not realised by an H -equivalence (one that preserves the multiplicative structures of the spaces involved). \square

Corollary 2.2. *Let G be a compact connected simple Lie group and S a toral subgroup in G . Then $H_*(\Omega(G/S); \mathbb{Z})$ is torsion free.*

Proof. In [7] the authors showed that if T is a maximal torus in a compact connected Lie group then $H_*(\Omega(G/T); \mathbb{Z})$ is torsion free. For G simply connected, G/S is simply connected as well and the statement follows by Proposition 2.1.

The only compact connected simple Lie group which is not simply connected is $SO(n)$. We show that $SO(n)/T^k$, where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ is simply connected by induction. There is a fibration sequence

$$SO(2k)/T^k \longrightarrow SO(n)/T^k \longrightarrow SO(n)/SO(2k).$$

Since $\pi_i(V_{n,2k}) = 0$ for $0 \leq i \leq 2k$ and $SO(2k)/T^k$ is simply connected (see for example [13], [7]), we conclude that $SO(n)/T^k$ is simply connected. Thus the statement for $\Omega(SO(n)/S)$ holds. \square

3. THE RATIONAL HOMOLOGY OF LOOP SPACES

In this section we compute rational homology of the based loop space on generalised symmetric space G/S of a simple compact Lie group G having toral isotropy subgroup S and zero Euler characteristic. In these generalised symmetric spaces, S is a non-maximal toral subgroup, that is, $\text{rank } S < \text{rank } G$, which can be obtained as the stationary subgroup of an outer automorphism of the group G . These generalised symmetric spaces are listed in [18] as: $SU(2n+1)/T^n$, $SU(2n)/T^n$, $SO(2n+2)/T^n$, $SO(8)/T^2$, and E_6/T^4 . They

are obtained by the outer automorphisms of the groups $SU(2n+1)$, $SU(2n)$, $SO(2n+2)$, $SO(8)$ and E_6 having respectively the orders $2n+1$, $2n-1$, $2n$, 12 and 18 .

3.1. The rational cohomology of homogeneous spaces. We recall some classical results on the cohomology of homogeneous spaces with rational coefficients. Consider a homogeneous space G/H , where G is a compact connected Lie group and H its connected closed subgroup. The Hopf theorem [3] states that the rational cohomology algebra of G is an exterior algebra

$$H^*(G, \mathbb{Q}) \cong \wedge(z_1, \dots, z_n)$$

where z_1, \dots, z_n are the universal transgressive generators in degree $\deg z_i = 2k_i - 1$, where k_1, \dots, k_n are the exponents of the group G (see [6]) and $n = \text{rank } G$ denotes the dimension of the maximal torus in G . Recall [3] that for a fibration $F \rightarrow E \rightarrow B$, we can define a map τ called transgression which maps a subgroup of $H^*(F)$ to a quotient of $H^*(B)$. If we consider the universal G -bundle $G \rightarrow EG \rightarrow BG$, then elements $x \in H^*(G)$ on which the transgression is defined are said to be universal transgressive.

The rational cohomology algebra of the classifying space BG of a group G is the algebra of polynomials on the maximal abelian subalgebra \mathfrak{t} of the Lie algebra \mathfrak{g} for G which are invariant under the action of the Weyl group W_G , that is,

$$H^*(BG, \mathbb{Q}) \cong \mathbb{Q}[\mathfrak{t}]^{W_G}.$$

This algebra is generated by the polynomials P_1, \dots, P_n called Weyl invariant generators, which correspond to z_1, \dots, z_n by transgression in the universal bundle for G and thus have degree $\deg P_i = 2k_i$, $1 \leq i \leq n$ (see [3]).

Denote by $\mathfrak{s} \subset \mathfrak{t}$ the maximal abelian subalgebra of the Lie algebra \mathfrak{h} for H . Since $W_H \subset W_G$, the polynomials from $H^*(BG, \mathbb{Q})$ when restricted to \mathfrak{s} belong to $H^*(BH, \mathbb{Q})$.

The *Cartan algebra* for a homogeneous space G/H (see [3]) is a differential graded algebra (C, d) defined by:

$$C = H^*(BH, \mathbb{Q}) \otimes H^*(G, \mathbb{Q}), \quad d(b \otimes 1) = 0, \quad d(1 \otimes z_i) = \rho^*(P_i) \otimes 1$$

where $\rho^* : \mathbb{Q}[\mathfrak{t}]^{W_G} \rightarrow \mathbb{Q}[\mathfrak{s}]^{W_H}$ denotes the restriction.

By the famous Cartan theorem [3], the Cartan algebra determines the rational cohomology of G/H , that is,

$$H^*(G/H, \mathbb{Q}) \cong H^*(C, d).$$

There is a wide class of homogeneous spaces, called Cartan pair homogeneous spaces in terminology of [9] or normal position homogeneous spaces in terminology of [11], which behave nicely from the point of view of both rational cohomology and rational homotopy theory. A homogeneous space G/H is a Cartan pair homogeneous space if one can choose n algebraically independent generators $P_1, \dots, P_n \in \mathbb{Q}[\mathfrak{t}]^{W_G}$ such that $\rho^*(P_{r+1}), \dots, \rho^*(P_n)$ belong to the ideal in $\mathbb{Q}[\mathfrak{s}]^{W_H}$ generated by $\rho^*(P_1), \dots, \rho^*(P_r)$. Furthermore, when this is the case one can choose P_{r+1}, \dots, P_n such that $\rho^*(P_{r+1}) = \dots = \rho^*(P_n) = 0$. Then the Cartan theorem directly implies that the rational cohomology algebra for these spaces is given by

$$(1) \quad H^*(G/H; \mathbb{Q}) \cong H^*(BH, \mathbb{Q}) / \langle \rho^*(P_1), \dots, \rho^*(P_r) \rangle \otimes \wedge(z_{r+1}, \dots, z_n).$$

In this case the sequence $\rho^*(P_1), \dots, \rho^*(P_r)$ is regular, that is, the class $[\rho^*(P_i)]$ in $H^*(BH; \mathbb{Q}) / \langle \rho^*(P_1), \dots, \rho^*(P_{i-1}) \rangle$ is not a zero divisor. We can further assume that algebra (1) is reduced. Thus we can consider only those elements of the regular sequence which are decomposable, eliminating the generators in $H^*(BH, \mathbb{Q})$ which are linear combinations of $\rho^*(P_1), \dots, \rho^*(P_r)$.

Among examples of Cartan pair homogeneous spaces are homogeneous spaces of positive Euler characteristic, compact symmetric spaces [3], [17] and generalised symmetric spaces [18].

3.2. On minimal model theory. (see [8]) Let (A, d_A) be a commutative graded differential algebra over the real numbers. A differential graded algebra (μ_A, d) is called *minimal model* for (A, d_A) if

- (i) there exists differential graded algebra morphism $h_A: (\mu_A, d) \rightarrow (A, d_A)$ inducing an isomorphism in their cohomology algebras (such h_A is called quasi-isomorphism);
- (ii) (μ_A, d) is a free algebra in the sense that $\mu_A = \wedge V$ is an exterior algebra over graded vector space V ;
- (iii) differential d is indecomposable meaning that for a fixed set $V = \{P_\alpha, \alpha \in I\}$ of free generators of μ_A for any $P_\alpha \in V$, $d(P_\alpha)$ is a polynomial in generators P_β with no linear part.

Two algebras are said to be *weakly equivalent* if there exists quasi-isomorphism between them. This is equivalent to say that these algebras have isomorphic minimal models. The algebra (A, d_A) is said to be *formal* if it is weakly equivalent to the algebra $(H^*(A), 0)$.

For a smooth connected manifold M , the minimal model is by definition the minimal model of its de Rham algebra of differential forms $\Omega_{DR}(M)$. In the case when M is simply connected manifold its minimal model completely classifies its rational homotopy type. The manifold M is said to be formal (in the sense of Sullivan) if $\Omega_{DR}(M)$ is a formal algebra.

It is a classical result [15] that compact homogeneous space is formal if and only if it is a Cartan pair homogeneous space. In particular, all generalised symmetric spaces are formal in the sense of Sullivan [10]. Thus the minimal model for Cartan pair homogeneous space G/H (see [5]) is given by $\mu = (\wedge V, d)$ where

$$(2) \quad V = (u_1, \dots, u_l, v_1, \dots, v_k, z_{r+1}, \dots, z_n)$$

$$(3) \quad d(u_i) = 0, \quad d(v_j) = \rho^*(P_j), \quad d(z_i) = 0$$

with u_i correspond to the remaining generators of $H^*(BH; \mathbb{Q})$, while v_j correspond to the remaining elements of the sequence $\rho^*(P_1), \dots, \rho^*(P_r)$.

3.3. The loop space rational homology. Let $\mu = (\wedge V, d)$ be a Sullivan minimal model of a simply connected topological space M with the rational homology of finite type. Then $d: V \rightarrow \wedge^{\geq 2} V$ can be decomposed as $d = d_1 + d_2 + \dots$, where $d_i: V \rightarrow \wedge^{\geq i+1} V$. In particular, d_1 is called the *quadratic part* of the differential d .

The homotopy Lie algebra \mathcal{L} of μ is defined in the following way. The underlying graded vector space L is given by

$$sL = \text{Hom}(V, \mathbb{Q})$$

where sL denotes the usual suspension defined by $(sL)_i = (L)_{i-1}$. We can define a pairing $\langle ; \rangle: V \times sL \rightarrow \mathbb{Q}$ by $\langle v; sx \rangle = (-1)^{\deg v} sx(v)$ and extend it to $(k+1)$ -linear maps

$$\wedge^k V \times sL \times \dots \times sL \rightarrow \mathbb{Q}$$

by letting

$$\langle v_1 \wedge \dots \wedge v_k; sx_k, \dots, sx_1 \rangle = \sum_{\sigma \in S_k} \epsilon_\sigma \langle v_{\sigma(1)}; sx_1 \rangle \dots \langle v_{\sigma(k)}; sx_k \rangle$$

where S_k is the symmetric group and $\epsilon_\sigma = \pm 1$ are determined by

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \epsilon_\sigma v_1 \wedge \cdots \wedge v_k.$$

Then the space L inherits a Lie bracket $[\cdot, \cdot]: L \times L \longrightarrow L$ from d_1 uniquely determined by

$$(4) \quad \langle v; s[x, y] \rangle = (-1)^{\deg y + 1} \langle d_1 v; sx, sy \rangle \quad \text{for } x, y \in L, v \in V.$$

Denote by \mathcal{L} the Lie algebra $(L, [\cdot, \cdot])$.

On the other hand in the category of topological spaces and continuous maps, we can define the Samelson products $[f, g]: S^{p+q} \longrightarrow \Omega M$ of maps $f: S^p \longrightarrow \Omega M$ and $g: S^q \longrightarrow \Omega M$ by the composite

$$S^p \wedge S^q \xrightarrow{f \wedge g} \Omega M \wedge \Omega M \xrightarrow{c} \Omega M$$

where c is given by the multiplicative commutator, that is, $c(x, y) = x \cdot y \cdot x^{-1} \cdot y^{-1}$. Recall that there is a graded Lie algebra $L_M = (\pi_*(\Omega M) \otimes \mathbb{Q}; [\cdot, \cdot])$ called the *rational homotopy Lie algebra of M* , for which the commutator $[\cdot, \cdot]$ is given by the Samelson product. There is an isomorphism between the rational homotopy Lie algebra L_M and the homotopy Lie algebra \mathcal{L} of μ . Milnor and Moore [see Appendix in [12]] showed that for a path connected homotopy associative H -space with unit G , there is an isomorphism of Hopf algebras $U(\pi_*(G) \otimes \mathbb{Q}) \cong H_*(G; \mathbb{Q})$. As loop multiplication is homotopy associative with unit, applying the Milnor and Moore theorem to our case, it follows that

$$H_*(\Omega M; \mathbb{Q}) \cong U\mathcal{L}$$

where $U\mathcal{L}$ is the universal enveloping algebra for \mathcal{L} . Further on,

$$U\mathcal{L} \cong T(L) / \langle xy - (-1)^{\deg x \deg y} yx - [x, y] \rangle.$$

For a more detailed account of this construction see for example [8], Chapters 12 and 16.

3.4. The loop space of some generalised symmetric spaces. Recall from [18] that the rational cohomology of $SU(2n+1)/T^n$ is given by

$$H^*(SU(2n+1)/T^n; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_n] / \langle P_2, P_4, \dots, P_{2n} \rangle \otimes \wedge(z_3, z_5, \dots, z_{2n+1}),$$

where $P_{2i} = \sum_{j=1}^n x_j^{2i}$, $1 \leq i \leq n$ and $\deg x_j = 2$, $\deg z_{2j+1} = 4j+1$. Since $SU(2n+1)/T^n$ is formal, the minimal model for $SU(2n+1)/T^n$ is the minimal model for the commutative differential graded algebra $(H^*(M; \mathbb{Q}), d=0)$. It is given by $\mu = (\Lambda V, d)$, where

$$V = (x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n)$$

with $\deg u_i = 2$, $\deg y_i = 4i-1$ and $\deg z_i = 4i+1$ for $1 \leq i \leq n$.

The differential d is defined by

$$(5) \quad d(x_i) = d(z_i) = 0, \quad d(y_i) = \sum_{j=1}^n x_j^{2i}.$$

Theorem 3.1. *The rational homology ring of the loop space on the manifold $SU(2n+1)/T^n$ is*

$$(6) \quad H_*(\Omega(SU(2n+1)/T^n); \mathbb{Q}) \cong (T(a_1, \dots, a_n) / \langle a_1^2 = \dots = a_n^2, a_i a_j = -a_j a_i \mid 1 \leq i, j \leq n \rangle) \otimes \mathbb{Q}[b_2, \dots, b_n, c_1, \dots, c_n]$$

where the generators a_i are of degree 1 for $1 \leq i \leq n$, the generators b_j are of degree $4j-2$ for $2 \leq j \leq n$, and the generators c_k are of degree $4k$ for $1 \leq k \leq n$.

Proof. The underlying vector space of the homotopy Lie algebra \mathcal{L} of μ is given by

$$L = (a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n)$$

where $\deg(a_i) = 1$, $\deg(b_i) = 4i - 2$ and $\deg(c_i) = 4i$ for $1 \leq k \leq n$.

In order to define Lie brackets we need the quadratic part d_1 of the differential in the minimal model. In this case, using the differential d defined in (5), the quadratic part d_1 is given by

$$d_1(x_i) = d_1(z_i) = 0 \quad d_1(y_1) = 2 \sum_{j=1}^n x_j^2 d_1(y_j) = 0 \text{ for } k \neq 1.$$

By the defining property of the Lie bracket stated in (4), we have

$$\begin{aligned} \langle y_1, s[a_i, a_i] \rangle &= \left\langle \sum x_j^2; sa_i, sa_i \right\rangle = \langle x_i^2; sa_i, sa_i \rangle = 2 \\ [a_i, b_j] &= [a_i, c_j] = [b_i, b_j] = [b_i, c_j] = [c_i, c_j] = 0 \text{ for } 1 \leq i, j \leq n \text{ and} \\ [a_i, a_j] &= 0 \text{ for } i \neq j. \end{aligned}$$

Therefore in the tensor algebra $T(a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n)$, the Lie brackets above induce the following relations

$$\begin{aligned} a_i a_j + a_j a_i &= 0 && \text{for } 1 \leq i, j \leq n, i \neq j \\ a_i^2 &= b_1 && \text{for } 1 \leq i \leq n \\ a_i b_j &= b_j a_i && \text{for } 1 \leq i, j \leq n \\ a_i c_j &= c_j a_i && \text{for } 1 \leq i, j \leq n \\ b_i b_j &= b_j b_i && \text{for } 1 \leq i, j \leq n \\ b_i c_j &= c_j b_i && \text{for } 1 \leq i, j \leq n \\ c_i c_j &= c_j c_i && \text{for } 1 \leq i, j \leq n. \end{aligned}$$

This proves the theorem. \square

Since the rational cohomology algebra for $SU(2n)/T^n$ is given by

$$H^*(SU(2n)/T^n; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_n] / \langle P_2, P_4, \dots, P_{2n} \rangle \otimes \wedge(z_3, z_5, \dots, z_{2n-1}),$$

where $P_{2i} = \sum_{j=1}^n x_j^{2i}$, $1 \leq i \leq n$ and $\deg x_j = 2$, $\deg z_{2j+1} = 4j + 1$ for $1 \leq j \leq n - 1$ in the same way as in Theorem 3.1 we prove the following.

Theorem 3.2. *The rational homology of the loop space on the manifold $SU(2n)/T^n$ is*

$$(7) \quad H_*(\Omega(SU(2n)/T^n); \mathbb{Q}) \cong (T(a_1, \dots, a_n) / \langle a_1^2 = \dots = a_n^2, a_i a_j = -a_j a_i \mid 1 \leq i, j \leq n \rangle) \otimes \mathbb{Q}[b_2, \dots, b_n, c_1, \dots, c_{n-1}]$$

where the generators a_i are of degree 1 for $1 \leq i \leq n$, the generators b_j are of degree $4j - 2$ for $2 \leq j \leq n$, and the generators c_k are of degree $4k$ for $1 \leq k \leq n - 1$. \square

Theorem 3.3. *The rational homology of the based loop space on $SO(2n+2)/T^n$ is given by*

$$H_*(\Omega(SO(2n+2)/T^n); \mathbb{Q}) \cong \left(T(a_1, \dots, a_n) / \left\langle \begin{array}{l} a_1^2 = \dots = a_n^2, \\ a_k a_l = -a_l a_k \text{ for } k \neq l \end{array} \right\rangle \right) \otimes \mathbb{Q}[b_2, \dots, b_n, b_{n+1}]$$

where the generators a_i are of degree 1 for $1 \leq i \leq n$, the generators b_k are of degree $4k - 2$ for $2 \leq k \leq n$, and the generator b_{n+1} is of degree $2n$.

Proof. The rational cohomology of $SO(2n+2)/T^n$ is given by

$$H^*(SO(2n+2)/T^n; \mathbb{Q}) \cong (\mathbb{Q}[x_1, \dots, x_n] / \langle P_2, P_4, \dots, P_{2n} \rangle) \otimes \Lambda(z)$$

where $P_{2k} = \sum_{i=1}^n x_i^{2k}$, $\deg(x_i) = 2$ for $1 \leq i \leq n$, $\deg(P_{2k}) = 4k$ for $2 \leq k \leq n$, and $\deg(z) = 2n+1$. The minimal model for $SO(2n+2)/T^n$ is the minimal model for the commutative differential algebra $(H^*(SO(2n+2)/T^n; \mathbb{Q}), d=0)$. It is given by $\mu = (\Lambda V, d)$, where

$$V = (x_1, \dots, x_n, y_1, \dots, y_n, y_{n+1}),$$

$$\deg(x_k) = 2, \deg(y_k) = 4k-1 \text{ for } 1 \leq k \leq n \text{ and } \deg(y_{n+1}) = 2n+1.$$

The differential d is given by

$$(8) \quad d(x_k) = 0, \quad d(y_k) = P_{2k} = \sum_{i=1}^n x_i^{2k} \text{ for } 1 \leq k \leq n \text{ and } d(y_{n+1}) = 0.$$

Now the underlying vector space of the homotopy Lie algebra L of μ is given by

$$L = (a_1, \dots, a_n, b_1, \dots, b_n, b_{n+1})$$

where $\deg(a_k) = 1$, $\deg(b_k) = 4k-2$ for $1 \leq k \leq n$ and $\deg(b_{n+1}) = 2n$. In order to define Lie brackets we need the quadratic part d_1 of the differential d defined in (8). It is given by

$$d_1(x_k) = 0, \quad d_1(y_1) = \sum_{i=1}^n x_i^2, \quad d_1(y_k) = 0 \text{ for } 2 \leq k \leq n \text{ and } d_1(y_{n+1}) = 0.$$

For dimensional reasons, we have

$$[a_k, b_l] = [b_s, b_l] = 0 \text{ for } 1 \leq k \leq n \text{ and } 1 \leq s, l \leq n+1.$$

By the defining property of the Lie bracket stated in (4), we have

$$\langle y_1, s[a_k, a_k] \rangle = \left\langle \sum x_i^2; sa_k, sa_k \right\rangle = \langle x_k^2; sa_k, sa_k \rangle = 1 \text{ and}$$

$$\langle y_1, s[a_k, a_l] \rangle = 0 \text{ for } k \neq l$$

resulting in the non-trivial commutators

$$[a_k, a_k] = 2b_1 \text{ for } 1 \leq k \leq n.$$

Therefore in the tensor algebra $T(a_1, \dots, a_n, b_1, \dots, b_n, b_{n+1})$ the Lie brackets above induce the following relations

$$\begin{aligned} b_k b_l &= b_l b_k && \text{for } 1 \leq k, l \leq n+1 \\ 2a_k^2 &= b_1 && \text{for } 1 \leq k \leq n \\ a_k a_l &= -a_l a_k && \text{for } 1 \leq k, l \leq n \\ a_k b_l &= b_l a_k && \text{for } 1 \leq k \leq n \text{ and } 1 \leq l \leq n+1. \end{aligned}$$

Thus

$$(9) \quad U\mathcal{L} = \left(T(a_1, \dots, a_n) / \left\langle \begin{array}{l} a_1^2 = \dots = a_n^2, \\ a_k a_l = -a_l a_k \text{ for } k \neq l \end{array} \right\rangle \right) \otimes \mathbb{Q}[b_2, \dots, b_n, b_{n+1}].$$

This proves the theorem. \square

Theorem 3.4. *The rational homology ring of the loop space on the manifold $SO(8)/T^2$ is*

$$(10) \quad H_*(\Omega(SO(8)/T^2); \mathbb{Q}) \cong (T(a_1, a_2) / \langle a_1^2 = a_2^2 = a_1 a_2 + a_2 a_1 \rangle) \otimes \mathbb{Q}[b_2, c_1, c_2]$$

where the generators a_i are of degree 1 for $i = 1, 2$, the generators b_2 is of degree 10, and the generators c_1, c_2 are of degree 6.

Proof. The rational cohomology for $SO(8)/T^2$ (see for example [18]) is given by

$$H^*(SO(8)/T^2; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2] / \langle x_1^2 + x_2^2 + x_1 x_2, (x_1 + x_2)^2 x_1^2 x_2^2 \rangle \otimes \wedge(z_2, z_4)$$

implying that its minimal model is of the form $(\wedge V, d)$ where $V = (x_1, x_2, y_1, y_2, z_1, z_2)$, $\deg x_i = 2$, $\deg y_1 = 3$, $\deg y_2 = 11$, $\deg z_1 = \deg z_2 = 7$ and

$$d(x_i) = d(z_i) = 0, \quad d(y_1) = x_1^2 + x_2^2 + x_1 x_2, \quad d(y_2) = (x_1 + x_2)^2 x_1^2 x_2^2.$$

Therefore its rational homotopy Lie algebra is given by $L = (a_1, a_2, b_1, b_2, c_1, c_2)$, $\deg a_i = 1$, $\deg b_1 = 2$, $\deg b_2 = 10$, $\deg c_1 = \deg c_2 = 6$, and

$$[a_i, b_j] = [a_i, c_j] = [b_i, b_j] = [b_i, c_j] = [c_i, c_j] = 0, \quad [a_1, a_1] = [a_2, a_2] = 2b_1, \quad [a_1, a_2] = b_1.$$

These commutators imply the needed relations in UL and thus we obtain the ring structure of the rational homology for the based loop space on $SO(8)/T^2$. \square

Theorem 3.5. *The rational homology of the loop space on the manifold E_6/T^4 is given by*

$$(11) \quad H^*(E_6/T^4; \mathbb{Q}) \cong (T(a_1, a_2, a_3, a_4) / \langle a_1^2 = a_2^2 = a_3^2 = a_4^2, a_i a_j = -a_j a_i \text{ for } i \neq j \rangle) \otimes \mathbb{Q}[b_4, b_5, b_7, b_8, b_{11}],$$

where $\deg a_i = 1$, $1 \leq i \leq 4$ and $\deg b_j = 2j$, $j = 4, 5, 7, 8, 11$.

Proof. Let T^6 be the maximal torus in E_6 containing T^4 . The Weyl invariant generating polynomials for $H^*(BE_6; \mathbb{Q})$ may be taken to be $P_k = \sum_{i=1}^6 (x_i \pm \epsilon)^k + \sum_{1 \leq i < j \leq 6} (-x_i - x_j)^k$ for $k = 2, 5, 6, 8, 9, 12$. Here x_1, \dots, x_6 and ϵ denote the canonical coordinates for T^6 in E_6 where $\sum_{i=1}^6 x_i = 0$ (see for example [15]). The rational cohomology algebra for E_6/T^4 is in [19] given by

$$H^*(E_6/T^4; \mathbb{Q}) = H^*(BT^4; \mathbb{Q}) / \langle \rho^*(P_2), \rho^*(P_6), \rho^*(P_8), \rho^*(P_{12}) \rangle \otimes \wedge(z_5, z_9)$$

where ρ^* denotes the restriction from the maximal torus T^6 in E_6 to the torus T^4 . If H_i , $1 \leq i \leq 6$ is Shevalley basis for T^6 , then the basis for T^4 in the case of generalised symmetric space E_6/T^4 is by [18] given by $\bar{H}_1 = H_1 + H_5$, $\bar{H}_2 = H_2 + H_4$, $\bar{H}_3 = H_3$ and $\bar{H}_4 = H_6$. Therefore it follows from [19] that when restricted to T^4 , x_1, \dots, x_6 satisfy $x_1 = -x_6$, $x_2 = -x_5$, $x_3 = -x_4$.

Thus the minimal model for E_6/T^4 is given by

$$V = (u_1, u_2, u_3, u_4, v_1, v_4, v_5, v_7, v_8, v_{11})$$

where $\deg u_i = 2$, $\deg v_j = 2j + 1$, and $d(u_i) = 0$, $d(v_4) = d(v_8) = 0$ while $d(v_i) = \rho^*(P_{2i+2})$ for $i = 1, 5, 7, 11$. It further implies that the quadratic part d_1 of the differential d vanishes except $d_1(v_1) = 12(u_1^2 + u_2^2 + u_3^2 + u_4^2)$. Therefore for the homotopy Lie algebra of this minimal model we obtain

$$L = (a_1, a_2, a_3, a_4, b_1, b_4, b_5, b_7, b_8, b_{11})$$

where $\deg a_i = 1$, $\deg b_j = 2j$ and the brackets are given by

$$[a_i, a_j] = [a_i, b_j] = [b_k, b_l] = 0 \quad [a_i, a_i] = 24b_1.$$

It implies that $a_1^2 = a_2^2 = a_3^2 = a_4^2 = 12b_1$ and $a_i a_j + a_j a_i = 0$ for $i \neq j$. \square

4. INTEGRAL PONTRJAGIN HOMOLOGY

In this section we study the integral Pontrjagin ring structure of generalised symmetric spaces $\Omega(G/S)$, where G is a simple Lie group and S an appropriate toral subgroup whose rational homology we calculated in the previous section. In addition to the rational homology calculation we make use of the results from [4] and [14] on integral homology of the identity component $\Omega_0 G$ of the loop space on G . Recall that for a compact connected Lie group G , the Pontrjagin homology $H_*(\Omega_0 G; \mathbb{Q})$ is primitively generated, that is, it is generated as an algebra by its space of primitive elements. It is well known that if G is a simply connected Lie group, then $\pi_2(G/S) \cong \mathbb{Z}^{\dim S}$ and $\pi_3(G/S) \cong \mathbb{Z}$. Let $\alpha, \beta: S^2 \rightarrow G/S$ and let $\tilde{\alpha}, \tilde{\beta}: S^1 \rightarrow \Omega(G/S)$ denote their adjoints, respectively. The adjoint $W(\alpha, \beta)$ of the Samelson product $[\tilde{\alpha}, \tilde{\beta}]: S^2 \rightarrow \Omega(G/S)$ is called the Whitehead product of α and β . Let

$$W: \pi_2(G/S) \otimes \pi_2(G/S) \rightarrow \pi_3(G/S)$$

denote also the pairing given by the Whitehead product. In what follows, we identify $H_1(S; \mathbb{Z})$ with $\pi_2(G/S)$ and $H_2(\Omega G; \mathbb{Z})$ with $\pi_3(G/S)$ via natural homomorphisms. Thus since there is no torsion in homology, and using the rational homology results from Section 3, we obtain that there is a split extension of algebras

$$1 \longrightarrow H_*(\Omega G; \mathbb{Z}) \longrightarrow H_*(\Omega(G/S); \mathbb{Z}) \longrightarrow H_*(S; \mathbb{Z}) \longrightarrow 1$$

with the extension given by $[\alpha, \beta] = W(\alpha, \beta) \in H_2(\Omega G; \mathbb{Z})$, where $\alpha, \beta \in H_1(S; \mathbb{Z})$.

4.1. The integral homology of $\Omega(SU(2n+1)/T^n)$ and $\Omega(SU(2n)/T^n)$.

Theorem 4.1. *The integral Pontrjagin homology ring of the loop space on $SU(2n+1)/T^n$ is*

$$H_*(\Omega(SU(2n+1)/T^n); \mathbb{Z}) \cong (T(x_1, \dots, x_n) \otimes \mathbb{Z}[y_2, \dots, y_n, z_1, \dots, z_n]) / \langle x_1^2 = \dots = x_n^2, x_p x_q + x_q x_p \text{ for } 1 \leq p, q \leq n, p \neq q \rangle$$

where the generators x_1, \dots, x_n are of degree 1, the generators y_i are of degree $4i - 2$ for $2 \leq i \leq n$, the generators z_i are of degree $4i$ for $1 \leq i \leq n$.

Proof. We explain the extension of the algebra in more detail for the generalised symmetric space $SU(2n+1)/T^n$. Notice that there is a monomorphism of two split extensions of algebras

$$\begin{array}{ccccccc} 1 & \longrightarrow & H_*(\Omega SU(2n+1); \mathbb{Z}) & \longrightarrow & H_*(\Omega(SU(2n+1)/T^n); \mathbb{Z}) & \longrightarrow & H_*(T^n; \mathbb{Z}) \longrightarrow 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & H_*(\Omega SU(2n+1); \mathbb{Q}) & \longrightarrow & H_*(\Omega(SU(2n+1)/T^n); \mathbb{Q}) & \longrightarrow & H_*(T^n; \mathbb{Q}) \longrightarrow 1. \end{array}$$

Denote by $\bar{c}_2, \dots, \bar{c}_{2n+1}$ the universal transgressive generators in $H^*(SU(2n+1); \mathbb{Z})$ which map to the symmetric polynomials $c_2 = \sum_{1 \leq i < j \leq 2n+1} \hat{x}_i \hat{x}_j, \dots, c_{2n+1} = \hat{x}_1 \cdots \hat{x}_{2n} \hat{x}_{2n+1}$

generating $H^*(BSU(2n+1); \mathbb{Z})$. Let T^{2n} be a maximal torus which contains T^n . The elements $\hat{x}_1, \dots, \hat{x}_{2n}, \hat{x}_{2n+1}$ are the integral generators of $H_*(T^{2n}; \mathbb{Z})$ and $\sum_{i=1}^{2n+1} x_i = 0$.

Recall from [19] that $\hat{x}_{2i+1} = 0$ when restricted to T^n so that \hat{x}_{2i} are integral generators in $H^*(T^n; \mathbb{Z})$, where $1 \leq i \leq n$. We denote these generators \hat{x}_{2i} by x_i , where $1 \leq i \leq n$.

Now let y_1, \dots, y_{2n} be the integral generators of $H_*(\Omega SU(2n+1); \mathbb{Z})$ obtained by the transgression of the elements from $H_*(SU(2n+1); \mathbb{Z})$ which are the duals of $\bar{c}_2, \dots, \bar{c}_{2n+1}$. Further, the set of primitive elements in $H_*(\Omega SU(2n+1); \mathbb{Z})$ is spanned by the elements $\sigma_1, \dots, \sigma_{2n}$ which can be expressed in terms of y_1, \dots, y_{2n} using the Newton formula

$$(12) \quad \sigma_k = \sum_{i=1}^{k-1} (-1)^{i-1} \sigma_{k-i} y_i + (-1)^{k-1} k y_k, \quad 1 \leq k \leq 2n.$$

The integral elements $\sigma_1, \dots, \sigma_{2n}$ rationalise to the elements $b_1, c_1, \dots, b_n, c_n \in H_*(\Omega SU(2n+1); \mathbb{Q})$. The generators a_1, \dots, a_n in $H_*(T^n; \mathbb{Q})$ are the rationalised images of the integral generators x_1, \dots, x_n in $H_*(T^n; \mathbb{Z})$. To decide the integral extension, we consider the rational Pontrjagin ring structure (6) of $\Omega(SU(2n+1)/T^n)$. Looking at the above commutative diagram of the algebra extensions, we conclude that the integral elements

$$\begin{aligned} x_k x_l + x_l x_k & \quad \text{for } 1 \leq k, l \leq n, k \neq l, \\ x_k^2 - \sigma_1 & \quad \text{for } 1 \leq k \leq n, \\ x_k \sigma_l - \sigma_l x_k & \quad \text{for } 1 \leq k \leq n, 1 \leq l \leq 2n \\ \sigma_k \sigma_l - \sigma_l \sigma_k & \quad \text{for } 1 \leq k, l \leq 2n \end{aligned}$$

from $H_*(\Omega(SU(2n+1)/T^n); \mathbb{Z})$ map to zero in $H_*(\Omega(SU(2n+1)/T^n); \mathbb{Q})$. As the map between the algebra extensions is a monomorphism, we conclude that these integral elements are zero. Using that there is no torsion in homology and Newton formula (12), we have

$$\begin{aligned} x_k x_l + x_l x_k &= 0 & \text{for } 1 \leq k, l \leq n, k \neq l, \\ x_k^2 &= y_1 & \text{for } 1 \leq k \leq n, \\ x_k y_l - y_l x_k &= 0 & \text{for } 1 \leq k, l \leq n, \\ x_k z_l - z_l x_k &= 0 & \text{for } 1 \leq k, l \leq n, \\ y_k y_l - y_l y_k &= 0 & \text{for } 1 \leq k, l \leq n, \\ z_k y_l - y_l z_k &= 0 & \text{for } 1 \leq k, l \leq n, \\ z_k z_l - z_l z_k &= 0 & \text{for } 1 \leq k, l \leq n \end{aligned}$$

which completely describes the integral Pontrjagin ring of $\Omega(SU(2n+1)/T^n)$. \square

In the analogous way we have the following result.

Theorem 4.2. *The integral Pontrjagin homology ring of the loop space on $SU(2n)/T^n$ is*

$$H_*(\Omega(SU(2n)/T^n); \mathbb{Z}) \cong$$

$$(T(x_1, \dots, x_n) \otimes \mathbb{Z}[y_2, \dots, y_n, z_1, \dots, z_{n-1}]) / \langle x_1^2 = \dots = x_n^2, x_p x_q + x_q x_p \text{ for } 1 \leq p, q \leq n, p \neq q \rangle$$

where the generators x_1, \dots, x_n are of degree 1, the generators y_i are of degree $4i - 2$ for $2 \leq i \leq n$, and the generators z_i are of degree $4i$ for $1 \leq i \leq n - 1$. \square

4.2. The integral homology of $\Omega(SO(2n+2)/T^n)$ and $\Omega(SO(8)/T^2)$.

Lemma 4.3. *For $k \leq \text{rank } SO(m)$,*

$$Spin(m)/T^k \simeq SO(m)/T^k.$$

Proof. Consider the extended diagram of fibrations and cofibrations

$$\begin{array}{ccccc}
 F & \longrightarrow & S^1 & \longrightarrow & S^1 \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbb{Z}/2 & \longrightarrow & Spin(m) & \longrightarrow & SO(m) \\
 \downarrow & & \downarrow & & \downarrow \\
 K & \longrightarrow & Spin(m)/S^1 & \longrightarrow & SO(m)/S^1
 \end{array}$$

where F and K are the homotopy fibers of the corresponding horizontal maps. Since $SO(m)/T^k$ is simply connected for any $k \leq \text{rank } SO(m)$, we conclude that $F \simeq \mathbb{Z}/2$ and thus $K \simeq *$ proving that $Spin(m)/S^1 \simeq SO(m)/S^1$ and from there that $Spin(m)/T^k \simeq SO(m)/T^k$ for $k \leq \text{rank } SO(m)$. \square

Theorem 4.4. *The integral Pontrjagin homology ring of the loop space on $SO(2n+2)/T^n$ is*

$$\begin{aligned}
 & H_*(\Omega(SO(2n+2)/T^n); \mathbb{Z}) \cong \\
 & (T(x_1, \dots, x_n) \otimes \mathbb{Z}[y_1, \dots, y_{n-1}, y_n + z, y_n - z, 2y_{n+1}, \dots, 2y_{2n}]) / I
 \end{aligned}$$

where I is generated by

$$\begin{aligned}
 & x_1^2 - y_1, x_i^2 - x_{i+1}^2 \text{ for } 1 \leq i \leq n-1 \\
 & x_k x_l + x_l x_k \text{ for } k \neq l \\
 & y_i^2 - 2y_{i-1}y_{i+1} + 2y_{i-2}y_{i+2} - \dots \pm 2y_{2i} \text{ for } 1 \leq i \leq n-1 \\
 & (y_n + z)(y_n - z) - 2y_n y_{n+2} + \dots \pm 2y_{2n}
 \end{aligned}$$

where $\deg x_i = 1$ for $1 \leq i \leq n$, $\deg y_i = 2i$ for $1 \leq i \leq n-1$, $\deg(y_n + z) = \deg(y_n - z) = 2n$, $\deg 2y_i = 2i$ for $n+1 \leq i \leq 2n$ and $y_0 = 1$.

Proof. Since $SO(2n+2)/T^n \simeq Spin(2n+2)/T^n$ by Lemma 4.3, we have that $\Omega(SO(2n+2)/T^n) \cong \Omega(Spin(2n+2)/T^n)$. It is known that $\Omega Spin(2n+2) \cong \Omega_0 SO(2n+2)$, see for example [13].

Recall from [4] that the algebra $H_*(\Omega_0 SO(2n+2); \mathbb{Z})$ is generated by the elements $y_1, \dots, y_{n-1}, y_n + z, y_n - z, 2y_{n+1}, \dots, 2y_{2n}$ which satisfy the relations

$$\begin{aligned}
 & y_i^2 - 2y_{i-1}y_{i+1} + 2y_{i-2}y_{i+2} - \dots \pm 2y_{2i} = 0 \text{ for } 1 \leq i \leq n-1, \\
 & (y_n + z)(y_n - z) - 2y_n y_{n+2} + \dots \pm 2y_{2n} = 0.
 \end{aligned}$$

These relations eliminate $2y_{2i}$ as generators for $\lfloor \frac{n+2}{2} \rfloor \leq i \leq n-1$, while for $1 \leq i \leq \lfloor \frac{n+2}{2} \rfloor - 1$, they induce new relations on y_{2i} implying that y_{2i} are generators only in the homology of $\Omega_0 SO(2n+2)$ with coefficients where 2 is not invertible. The subspace of primitive elements in $H_*(\Omega_0 SO(2n+2); \mathbb{Z})$ is spanned by the elements $p_1, p_3, \dots, p_{n-1}, 2z, 2p_{n+1}, \dots, 2p_{2n-1}$ for n odd and by the elements $p_1, p_3, \dots, p_n, 2z, 2p_{n+2}, \dots, 2p_{2(n-1)+1}$ for $n+1$ even. These primitive generators are obtained by transgressing the elements in $H_*(SO(2n+2); \mathbb{Z})$ which are the Poincare duals of the universal transgressive generators $\bar{\sigma}_1, \dots, \bar{\sigma}_n, \bar{\lambda}$ in $H^*(SO(2n+2); \mathbb{Z})$. The generators $\bar{\sigma}_1, \dots, \bar{\sigma}_n, \bar{\lambda}$ map to the polynomials $\sigma_i(x_1^2, \dots, x_{n+1}^2)$ for $1 \leq i \leq n$ and $\lambda = x_1 \cdots x_{n+1}$ which generate the free part in $H^*(BSO(2n+2); \mathbb{Z})$. here x_1, \dots, x_n are the generators for $H_*(T^{n+1}; \mathbb{Z})$ where T^{n+1} is the maximal torus for $SO(2n+2)$. Recall from [18] that in the case of generalised symmetric space $SO(2n+2)/T^n$ we have that T^n is embedded in T^{n+1} in a such a way that $x_{n+1} = 0$ on T^n and thus gives the morphism $H_*(\Omega_0 SO(2n+2); \mathbb{Q}) \rightarrow H_*(\Omega(SO(2n+2)/T^n); \mathbb{Q})$.

The proof of the theorem is now analogue to the proof of Theorems 4.3 and 4.4 in [7]. We consider the morphism of two extensions of algebras

$$\begin{array}{ccccc} H_*(\Omega_0 SO(2n+2); \mathbb{Z}) & \longrightarrow & H_*(\Omega(SO(2n+2)/T^n); \mathbb{Z}) & \longrightarrow & H_*(T^n; \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ H_*(\Omega_0 SO(2n+2); \mathbb{Q}) & \longrightarrow & H_*(\Omega(SO(2n+2)/T^n); \mathbb{Q}) & \longrightarrow & H_*(T^n; \mathbb{Q}). \end{array}$$

and taking into account the rational homology calculations of $\Omega(SO(2n+2)/T^n)$ in Theorem 3.3 and the above description of the algebra $H_*(\Omega_0 SO(2n+2); \mathbb{Z})$ we come to the result. \square

Theorem 4.5. *The integral Pontrjagin homology ring of the loop space on $SO(8)/T^2$ is*

$$H_*(\Omega(SO(8)/T^2); \mathbb{Z}) \cong (T(x_1, x_2) \otimes \mathbb{Z}[y_1, y_2, y_3 + z, y_3 - z, 2y_4, 2y_5, 2y_6]) / I$$

where I is generated by

$$\begin{aligned} & x_1^2 - y_1, x_1^2 - x_2^2 \\ & x_1^2 - x_1 x_2 + x_2 x_1 \\ & y_i^2 - 2y_{i-1}y_{i+1} + 2y_{i-2}y_{i+2} - \dots \pm 2y_{2i} \text{ for } 1 \leq i \leq 2 \\ & (y_3 + z)(y_3 - z) - 2y_3 y_5 + 2y_6 \end{aligned}$$

where $\deg x_1 = \deg x_2 = 1$, $\deg y_1 = 2$, $\deg y_2 = 4$, $\deg(y_3 + z) = \deg(y_3 - z) = 6$, $\deg 2y_i = 2i$ for $4 \leq i \leq 6$ and $y_0 = 1$. \square

Remark 4.6. It follows from [18] that for this generalised symmetric space T^2 is embedded in the maximal torus T^4 for $SO(8)$ in such a way that the canonical integral generators x_1, x_2, x_3, x_4 for $H_*(T^4; \mathbb{Z})$ restrict on T^2 to $x_4 = 0$ and $x_1 = x_2 + x_3$. We took this into account when we considered the morphism $H_*(\Omega_0 SO(8); \mathbb{Q}) \rightarrow H_*(\Omega(SO(8)/T^2); \mathbb{Q})$.

4.3. The integral homology of $\Omega(E_6/T^4)$. The integral homology algebra $H_*(\Omega E_6; \mathbb{Z})$ is described in [14] and it is given by

$$H_*(\Omega E_6; \mathbb{Z}) \cong \mathbb{Z}[y_1, y_2, y_3, y_4, y_5, y_7, y_8, y_{11}] / \langle y_1^2 - 2y_2, y_1 y_2 - 3y_3 \rangle$$

where $\deg y_i = 2i$ for $i = 1, 2, 3, 4, 5, 7, 8, 11$.

Using the same argument as for the previous cases, we deduce the integral Pontrjagin homology of the based loop space on E_6/T^4 .

Theorem 4.7. *The integral Pontrjagin homology ring of $\Omega(E_6/T^4)$ is given by*

$$H_*(\Omega(E_6/T^4); \mathbb{Z}) \cong$$

$$(T(x_1, x_2, x_3, x_4) \otimes \mathbb{Z}[y_1, y_2, y_3, y_4, y_5, y_7, y_8, y_{11}]) / I$$

where $I = \langle x_k^2 = 12y_1 \text{ for } 1 \leq k \leq 4, x_p x_q + x_q x_p \text{ for } 1 \leq p, q \leq 4, 2y_2 = x_1^4, 3y_3 = x_1^2 y_2 \rangle$ and where $\deg x_i = 1$ for $1 \leq i \leq 4$, and $\deg y_i = 2i$ for $i = 1, 2, 3, 4, 5, 7, 8, 11$.

5. APPENDIX

In the Appendix we illustrate that our methods could be applied not only to generalised symmetric spaces having toral stationary subgroup but wider. We concentrate on the example of $U(n)/(T^k \times U(n-k))$ for $k \leq n-2$.

Lemma 5.1. *The homology groups $H_*(\Omega(U(n)/T^l \times U(k)); \mathbb{Z})$, $l+k \leq n$, are torsion free.*

Proof. As $U(n)/U(k)$ is simply connected and $H_*(\Omega(U(n)/U(k)); \mathbb{Z}) \cong \mathbb{Z}[x_{2k}, \dots, x_{2(n-1)}]$, the statement follows by Proposition 2.1. \square

Theorem 5.2. *The rational homology ring of the loop space on the manifold $U(n)/(T^k \times U(n-k))$, $k \leq n-2$ is*

$$(13) \quad H_*(\Omega(U(n)/(T^k \times U(n-k))); \mathbb{Q}) \cong \wedge(a_1, \dots, a_k) \otimes \mathbb{Q}[b_{n-k+1}, \dots, b_n],$$

where the generators a_i are of degree 1 for $1 \leq i \leq k$, and the generators b_j are of degree $2j-2$ for $n-k+1 \leq j \leq n$.

Proof. It follows from [18] and [19] that the rational cohomology for this space is given by

$$H^*(U(n)/(T^k \times U(n-k)); \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_k] / \langle P_{n-k+1}, \dots, P_n \rangle$$

where $P_j = \sum_{i=1}^j x_i^j$ for $n-k+1 \leq j \leq n$. Since $\deg(P_j) = 2j \geq 2(n-k+1) \geq 6$ we conclude that the differential of the minimal model for this spaces has no quadratic part. Therefore the given procedure for the computation of the corresponding loop space homology leads to formula (13). \square

Theorem 5.3. *The integral Pontrjagin homology ring of the loop space on $U(n)/(T^k \times U(n-k))$, $k \leq n-2$ is*

$$(14) \quad H_*(\Omega(U(n)/(T^k \times U(n-k))); \mathbb{Z}) \cong \wedge(x_1, \dots, x_k) \otimes \mathbb{Z}[y_{n-k+1}, \dots, y_n]$$

where the generators x_i are of degree 1 for $1 \leq i \leq k$ and the generators y_j are of degree $2j-2$ for $n-k+1 \leq j \leq n$.

Proof. It follows from Lemma 5.1 and rational computation given in Theorem 5.2 that the fibration $\Omega(U(n)/U(n-k)) \rightarrow \Omega(U(n)/(T^k \times U(n-k))) \rightarrow T^k$ induces the monomorphism of two split algebra extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_*(\Omega(U(n)/U(k)); \mathbb{Z}) & \longrightarrow & H_*(\Omega(U(n)/(T^k \times U(n-k))); \mathbb{Z}) & \longrightarrow & H_*(T^k; \mathbb{Z}) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & H_*(\Omega(U(n)/U(k)); \mathbb{Q}) & \longrightarrow & H_*(\Omega(U(n)/(T^k \times U(n-k))); \mathbb{Q}) & \longrightarrow & H_*(T^k; \mathbb{Q}) & \longrightarrow & 1. \end{array}$$

As all the Whitehead products which define the extension are trivial, the theorem follows at once. \square

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SCHOOL OF MATHEMATICS, UNIVERSITY OF MANCHESTER, MANCHESTER M13 9PL, UNITED KINGDOM

FACULTY OF SCIENCE, UNIVERSITY OF MONTENEGRO, DŽORDŽA VAŠINGTONA BB, 81000 PODGORICA, MONTENEGRO

E-mail address: Jelena.Grbic@manchester.ac.uk

E-mail address: sterzic@ac.me