UNIVERSAL SPACES OF TWO-CELL COMPLEXES AND THEIR EXPONENT BOUNDS

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ABSTRACT. Let P^{2n+1} be a two-cell complex which is formed by attaching a $(2n+1)$–cell to a $2m$–sphere by a suspension map. We construct a universal space U for P^{2n+1} in the category of homotopy associative, homotopy commutative H–spaces. By universal we mean that U is homotopy associative, homotopy commutative, and has the property that any map $f\colon P^{2n+1} \longrightarrow Y$ to a homotopy associative, homotopy commutative H–space Y extends to a uniquely determined H–map $\overline{f}\colon U \longrightarrow Y$. We then prove upper and lower bounds of the H–homotopy exponent of U. In the case of a mod p^r Moore space U is the homotopy fibre $S^{2n+1}\{p^r\}$ of the p^r–power map on S^{2n+1}, and we reproduce Neisendorfer's result that $S^{2n+1}\{p^r\}$ is homotopy associative, homotopy commutative and that the p^r–power map on $S^{2n+1}\{p^r\}$ is null homotopic.

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1. INTRODUCTION

Free or universal objects have been of interest to mathematicians in many mathematical disciplines. In homotopy theory one of the first universal spaces is given in the category of homotopy associative H–spaces by the James construction (cf. [8]). The analogue of the James construction on a topological connected space X in the category of homotopy associative, homotopy commutative p–localised H–spaces is given by the following definition.

Definition 1.1. *Localised at p a universal space U_X of a topological space X is a homotopy associative, homotopy commutative H–space together with a map $i\colon X \longrightarrow U_X$ such that the following property holds:*

> *if Y is a homotopy associative, homotopy commutative localised at p H–space and $f\colon X \longrightarrow Y$ is any map, then f extends to a unique H–map $\overline{f}\colon U_X \longrightarrow Y$.*

This whole concept of studying universal spaces in the category of homotopy associative, homotopy commutative H–spaces is due to Gray (cf. [4]).

Despite the potential applications of universal spaces U_X in homotopy theory, there are as yet just a few known examples of these spaces.

As listed in [7], universal spaces are known to exist in the following cases. For one-cell complexes, spheres; for two-cell complexes, Moore spaces and the $(2np - 2)$–skeleton K of $\Omega^2 S^{2n+1}$; and for a three-cell complex L which is $(2np - 1)$–skeleton of $\Omega^2 S^{2n+1}$ are known.

The objective of this paper is to add to the list of examples a family of universal spaces of certain two-cell complexes. We construct these universal spaces using the method developed in [7] and rely heavily on Gray's decomposition of loop spaces on certain two-cell complexes [6]. The basic underlying reason why the methods given in [6], [7] work is the existence of a differential Lie algebra structure on homotopy groups with coefficients.

Our standing assumptions are that all spaces have a non-degenerate basepoint, are simply connected, have the homotopy type of a CW–complex, and are localised at an odd prime $p > 3$. Unless otherwise indicated, the ring of homology coefficients will be $\mathbb{Z}/p\mathbb{Z}$ and $H_*(X; \mathbb{Z}/p\mathbb{Z})$ will be written as $H_*(X)$.

The main object of consideration is given as follows.

Definition 1.2. *For $n \geq m$, let $\Theta \colon S^{2n-1} \longrightarrow S^{2m-1}$ be a given map. Then the two-cell complex P is defined as the homotopy cofibre of Θ.*

We decorate P with a superscript to denote the dimension of the top cell. Namely, $P^r = \Sigma^{r-2n} P$ for $r \geq 2n$.

Let U be the homotopy fibre of $\Sigma^2 \Theta \colon S^{2n+1} \longrightarrow S^{2m+1}$. Then our main theorem is:

Theorem 1.1. *U is the universal space of P^{2n+1}.*

The uniqueness assertion of an H–extension in Theorem 1.1 is powerful. It ensures that if a universal space exists then it and its H–structure that is homotopy associative and homotopy commutative are unique up to homotopy equivalence. The uniqueness of an H–extension can be also used to show that two H–maps $U \longrightarrow Y$ are homotopic by comparing their restrictions on P^{2n+1}. We use this to obtain exponent information about the spaces U in Theorem 1.1.

For an arbitrary space Y we define its *homotopy exponent*, denoted $\exp(Y) = p^t$, if t is the minimal power of p which annihilates the p–torsion in the homotopy groups of Y.

Theorem 1.2. *Let p^l be the order of $\Sigma \Theta$ and p^k the order of $\Sigma^2 \Theta$. Then*

$$p^{n-k} \leq \exp(U) \leq p^{n+l}.$$

As a special case of our work we reprove some results of Neisendorfer (cf. [10]) related to the mod p^r Moore space $P^{2n+1}(p^r)$. If the degree p^r map on S^{2n} is taken for $\Sigma \Theta$, then the corresponding two-cell complex is mod p^r Moore space $P^{2n+1}(p^r)$. According to Theorem 1.1, the universal space of $P^{2n+1}(p^r)$ is the homotopy fibre of $\Sigma^2 \Theta \colon S^{2n+1} \longrightarrow S^{2n+1}$.

As the degree p^r map and the p^r–power map on an odd dimensional sphere coincide, the universal space of $P^{2n+1}(p^r)$ is $S^{2n+1}\{p^r\}$, the homotopy fibre of the p^r–power map on S^{2n+1}. Thus $S^{2n+1}\{p^r\}$ is a homotopy associative, homotopy commutative H–space.

The mod–p H–*exponent* of an H–space X, denoted $H\exp(X)$, is p^k if k is the minimal power for which the pth–power map on X is null homotopic. We use the universal property of $S^{2n+1}\{p^r\}$ to find the H–exponent of $S^{2n+1}\{p^r\}$.

Proposition 1.3. $H\exp(S^{2n+1}\{p^r\}) = p^r$.

Neisendorfer's proofs of H–structure and H–exponent of $S^{2n+1}\{p^r\}$ involved turning homotopy fibrations into actual fibrations, and so were point set in nature. Using universal spaces, our proofs retain the flexibility of the homotopies and so are perhaps more transparent.

2. Preliminaries

In this section we recall some definitions and known results that will be used in subsequent sections.

Define the map $\mathrm{ad}\colon \bigvee_{i=0}^{\infty} P^{4n+3+2mi} \longrightarrow P^{2n+2}$ as the wedge sum of generalised Whitehead products given by

$$\mathrm{ad} = \bigvee_{i=0}^{\infty} \mathrm{ad}_i : P^{4n+3} \wedge S^{2mi} \longrightarrow P^{2n+2}$$

where $\mathrm{ad}_i = [\iota, \mathrm{ad}_{i-1}]$ and $\mathrm{ad}_0 = [1_{P^{2n+2}}, 1_{P^{2n+2}}] \circ \Sigma q$, while q is a right homotopy inverse to $P^{2n+1} \wedge P^{2n+1} \longrightarrow P^{4n+2}$ and $\iota\colon S^{2m+1} \longrightarrow P^{2n+2}$ is the inclusion of the bottom cell in P^{2n+2}. In [6, Theorem 1.2] Gray described the homotopy fibre of the map ad and gave a decomposition of ΩP^{2n+2}. Recall that U is the homotopy fibre of $\Sigma^2\Theta\colon S^{2n+1} \longrightarrow S^{2m+1}$.

Theorem 2.1. *There exists a homotopy fibration sequence:*
(1)
$$\Omega\Big(\bigvee_{i=0}^{\infty} P^{4n+3+2mi} \Big) \xrightarrow{\Omega\mathrm{ad}} \Omega P^{2n+2} \xrightarrow{\partial} U \xrightarrow{*} \bigvee_{i=0}^{\infty} P^{4n+3+2mi} \xrightarrow{\mathrm{ad}} P^{2n+2}$$

and a homotopy decomposition

(2)
$$\Omega P^{2n+2} \simeq U \times \Omega\Big(\bigvee_{i=0}^{\infty} P^{4n+3+2mi} \Big).$$

2.1. **Homotopy Groups with P–coefficients.** Homotopy groups with coefficients in P are defined as

$$\pi_k(X; P) = [P^k, X] \text{ for } k \geq 0.$$

Gray showed (cf. [5, Proposition 3.5]) that there is a Lie algebra structure on the homotopy groups with the coefficients in P given by mod P Samelson products. To begin, there is a splitting

(3) $$P^k \wedge P^l \simeq P^{k+l-2n+2m-1} \vee P^{k+l}.$$

Let $\mu_{k,l}^*: P^{k+l} \longrightarrow P^k \wedge P^l$ be the inclusion.

If G is a group-like space, and $\alpha \in \pi_k(G; P)$, $\beta \in \pi_l(G; P)$, then the *mod P Samelson product* $[\alpha, \ \beta]$ is defined as the composition

$$P^{k+l} \xrightarrow{\mu_{k,l}^*} P^k \wedge P^l \xrightarrow{\langle \alpha, \beta \rangle} G$$

where $\langle \alpha, \beta \rangle$ is the commutator $\langle \alpha, \beta \rangle = \alpha\beta\alpha^{-1}\beta^{-1}$ in G. *Mod P Whitehead products* are defined as the adjoints of mod P Samelson products. Gray showed that $\mu_{k,l}^*$ can be chosen so that the mod P Samelson product satisfies anti-symmetry and Jacobi identities.

Let $\sigma = 2n - 2m + 1$ be the dimensional gap between the top and the bottom cell of the two-cell complex P. Gray defined a Bockstein homomorphism

$$\beta: \pi_k(X; P) \longrightarrow \pi_{k-\sigma}(X; P)$$

for $k \geq 4n - 2m + 1$ that is a degree σ derivation with respect to the Lie bracket on the homotopy groups with coefficients in P. If X is a group-like space, $u \in \pi_k(X; P)$ and $v \in \pi_l(X; P)$, then

$$\beta[u, v] = [\beta u, v] + (-1)^k[u, \beta v].$$

The reduced homology $\widetilde{H}_*(P^k; \mathbb{Z}/p\mathbb{Z})$ is a free $\mathbb{Z}/p\mathbb{Z}$–module on two generators v and u in degrees k and $k - \sigma$, respectively. The Hurewicz map

$$h: \pi_k(X; P) \longrightarrow \widetilde{H}_k(X; \mathbb{Z}_{(p)}).$$

is given by $h(\alpha) = f_*(v)$, where f is in $\pi_k(X; P)$ and $f: P^k \longrightarrow X$ represents α. It is a Lie algebra homomorphism (cf. [5]) from the homotopy groups with P coefficients of X to the Lie algebra of primitives in $H_*(X; \mathbb{Z}/p\mathbb{Z})$, that is, if X is a group-like space and $\alpha \in \pi_*(X; P)$, $\beta \in \pi_*(X, P)$, then

(4) $$h([\alpha, \beta]) = [h(\alpha), h(\beta)].$$

2.2. **A homology decomposition of $\Omega\Sigma P^{2n+1}$.** Notice that $H_*(P^{2n+1})$ is the free $\mathbb{Z}/p\mathbb{Z}$–module with basis x, u in degrees $2m$ and $2n+1$. Since $H_*(P^{2n+1})$ is a trivial coalgebra, $H_*(\Omega\Sigma P^{2n+1})$ is the primitively generated tensor Hopf algebra $T(x, u)$ generated by x and u. Therefore $H_*(\Omega\Sigma P^{2n+1})$ can be considered as the universal enveloping Hopf algebra $U\mathcal{L}$ of the free Lie algebra $\mathcal{L} = \mathcal{L}\langle x, u \rangle$. Denote the commutator of $\mathcal{L}$ by $[\mathcal{L}, \mathcal{L}]$ and regard $\mathcal{L}_{ab}\langle x, u \rangle$ as the free graded abelian Lie algebra,

that is, as $\mathcal{L}/[\mathcal{L}, \mathcal{L}]$. The geometrical decomposition (2) implies there is an isomorphism in homology

$$(5) \qquad\qquad U\mathcal{L} \cong U[\mathcal{L}, \mathcal{L}] \otimes U\mathcal{L}_{ab}\langle x, u \rangle$$

of left $U[\mathcal{L}, \mathcal{L}]$–modules and right $U\mathcal{L}_{ab}\langle x, u \rangle$–comodules. A basis for $[\mathcal{L}, \mathcal{L}]$ is given in [6] by

$$(6) \qquad\quad W = \{y_k = \mathrm{ad}^k(x)[u, u] \text{ and } z_k = \mathrm{ad}^k(x)[x, u]\}.$$

Let $\widetilde{W}$ be the set of maps $\mathrm{ad}_k \colon P^{4n+3+2mk} \longrightarrow P^{2n+2}$ and their Bocksteins. Pair of basis elements (y_k, z_k) are the mod P Hurewicz images of the maps ad_k and their Bocksteins respectively. The H-map $\Omega(\bigvee_{i=0}^{\infty} P^{4n+3+2mi}) \xrightarrow{\Omega\mathrm{ad}} \Omega P^{2n+2}$ is uniquely determined by its restriction to $\bigvee_{i=0}^{\infty} P^{4n+2+2mi}$, and so the image of $(\Omega\mathrm{ad})_*$ is $U[\mathcal{L}, \mathcal{L}]$.

3. Homotopy Associativity and Homotopy Commutativity

To begin the argument proving the homotopy associativity and homotopy commutativity of U we recall a pair of constructions from classical homotopy theory.

Let X be a topological space, and i_L and i_R the inclusions of X into the wedge $X \vee X$. Looping, we can take the Samelson product $[\Omega i_L, \Omega i_R]$. Adjointing gives the Whitehead product $[\zeta_L, \zeta_R]$, where $\zeta_A = i_A \circ ev$ for $A = L, R$ and ev is the canonical evaluation map $ev \colon \Sigma\Omega X \longrightarrow X$. A classical result in homotopy theory asserts that there is a homotopy fibration

$$(7) \qquad\qquad \Sigma\Omega X \wedge \Omega X \xrightarrow{[\zeta_L, \zeta_R]} X \vee X \xrightarrow{i} X \times X$$

where i is the inclusion. The *universal Whitehead product of X* is defined as the composition

$$\Phi \colon \Sigma\Omega X \wedge \Omega X \xrightarrow{[\zeta_L, \zeta_R]} X \vee X \xrightarrow{\nabla} X$$

where ∇ is the fold map. Notice that any Whitehead product on X factors through the universal Whitehead product of X.

The *universal Samelson product of X* is defined as the adjoint of the universal Whitehead product of X, namely, as the commutator of the identity map on ΩX

$$[i_{\Omega X}, i_{\Omega X}] \colon \Omega X \wedge \Omega X \longrightarrow \Omega X.$$

The linchpin in showing that U is homotopy associative, homotopy commutative is the following theorem proved by Theriault (cf. [11]).

Theorem 3.1. *Let $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$ be a homotopy fibration sequence in which ∂ has a right homotopy inverse. Suppose that there*

is a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma\Omega B \wedge \Omega B & \longrightarrow & B \vee B \\ \downarrow & & \downarrow \nabla \\ E & \longrightarrow & B, \end{array}$$

where the upper composite in the square is the universal Whitehead product of B. Then the multiplication on F induced by the retraction off ΩB is both homotopy associative, homotopy commutative and the connecting map ∂ is an H–map.

Returning to our case, consider the fibration sequence

$$\Omega P^{2n+2} \xrightarrow{\partial} U \xrightarrow{*} \bigvee_{i=0}^{\infty} P^{4n+3+2mi} \xrightarrow{\mathrm{ad}} P^{2n+2}$$

of Theorem 2.1. Let $\Psi\colon \Sigma\Omega P^{2n+2} \wedge \Omega P^{2n+2} \longrightarrow P^{2n+2} \vee P^{2n+2}$ be the Whitehead product $[\zeta_L, \zeta_R]$ for $X = P$.

Lemma 3.2. *The Whitehead product*

$$\Psi\colon \Sigma\Omega P^{2n+2} \wedge \Omega P^{2n+2} \longrightarrow P^{2n+2} \vee P^{2n+2}$$

is homotopic to a sum of mod P Whitehead products.

Proof. Using James' theorem and splitting (3), $\Sigma\Omega P^{2n+2} \wedge \Omega P^{2n+2} \simeq \Sigma M$ for M a wedge of two-cell complexes P^k. Let θ be the restriction of $\Omega\Psi$ to M

$$\theta\colon M \xrightarrow{E} \Omega\Sigma M \xrightarrow{\simeq} \Omega(\Sigma\Omega P^{2n+2} \wedge \Omega P^{2n+2}) \xrightarrow{\Omega\Psi} \Omega(P^{2n+2} \vee P^{2n+2}).$$

Then θ is a Samelson product as it is the adjoint of the Whitehead product Ψ. Since $\Omega\Psi$ has a left homotopy inverse, as fibration (7) splits when looped, it is an inclusion in homology. Furthermore, the Hurewicz image of each summand P^k of M under the composite θ is a bracket in $\mathcal{L}\langle V \rangle$, where $H_*(\Omega(P^{2n+2} \vee P^{2n+2})) \cong U\mathcal{L}\langle V \rangle$ and $V = \widetilde{H}_*(P^{2n+1} \vee P^{2n+1})$. Using the identity and mod P Bockstein maps on each summand of $P^{2n+2} \vee P^{2n+2}$, it is clear that there exists a mod P Samelson product on $\Omega(P^{2n+2} \vee P^{2n+2})$ which has the same Hurewicz image as P^k. Summing these mod P Samelson products, one for each summand of M, gives a map $\lambda\colon M \longrightarrow \Omega(P^{2n+2} \vee P^{2n+2})$ with the property that $\lambda_* = \theta_*$. Each mod P Samelson product factors through the loop of the universal Whitehead product of P, as every Whitehead products on P factors through the universal Whitehead product of P. Therefore λ lifts to a map $\lambda'\colon M \longrightarrow \Omega(\Sigma\Omega P^{2n+2} \wedge \Omega P^{2n+2})$ with $\lambda \simeq \Omega\Psi \circ \lambda'$. Extend λ' to $\overline{\lambda}\colon \Omega\Sigma M \longrightarrow \Omega(\Sigma\Omega P \wedge \Omega P)$. As $\lambda_* = \theta_*$, we get $(\Omega\Psi \circ \overline{\lambda})_* = (\Omega\Psi)_*$. As $(\Omega\Psi)_*$ is a monomorphism, we must have $(\overline{\lambda})_*$ is an isomorphism. Hence $\overline{\lambda}$ is a homotopy equivalence. Taking adjoints then proves the Lemma. $\square$

By definition, the universal Whitehead product of P^{2n+2} is the composition $\Phi\colon \Sigma\Omega P^{2n+2} \wedge \Omega P^{2n+2} \xrightarrow{\Psi} P^{2n+2} \vee P^{2n+2} \xrightarrow{\nabla} P^{2n+2}$.

Lemma 3.3. *There is a lift*

$$
\begin{array}{ccc}
& & \Sigma\Omega P^{2n+2} \wedge \Omega P^{2n+2} \\
& \nearrow & \big\downarrow{\scriptstyle\Phi} \\
\bigvee_{i=0}^{\infty} P^{4n+3+2mi} \xrightarrow{\;\;\mathrm{ad}\;\;} & & P^{2n+2}
\end{array}
$$

of the universal Whitehead product Φ of P^{2n+2} to $\bigvee_{i=0}^{\infty} P^{4n+3+2mi}$.

Proof. Lemma 3.2 shows that the universal Whitehead product on P^{2n+2} is homotopic to a sum of mod P Whitehead products. The mod P Whitehead product defines the Lie bracket on the homotopy groups with P coefficients, endowing it with a Lie algebra structure. The set $\widetilde{W}$, defined after (6), consists of mod P Whitehead products which form a Lie basis for $[\mathcal{L}, \mathcal{L}]$. So the mod P Whitehead products from the universal Whitehead product can be rewritten as a linear combination of basis elements. $\qquad\square$

Theorem 3.4. *U is a homotopy associative, homotopy commutative H–space.*

Proof. Applying Theorem 3.1 to the fibration sequence

$$
\Omega P^{2n+2} \xrightarrow{\partial} U \xrightarrow{\;*\;} P^{4n+3} \rtimes \Omega S^{2m+1} \xrightarrow{\mathrm{ad}} P^{2n+2}
$$

and using Lemma 3.3, the Proposition follows. $\qquad\square$

4. A Universal Property of U

In this section we show that U satisfies the universal property in the category of homotopy associative, homotopy commutative H–spaces. Let $f\colon P^{2n+1} \longrightarrow Z$ be a map into a homotopy associative, homotopy commutative H–space. We show that there is a unique multiplicative extension $\overline{f}\colon U \longrightarrow Z$ of f.

Proposition 4.1. *Let $h\colon \Omega P^{2n+2} \longrightarrow Z$ be an H–map into a homotopy commutative H–space Z. Then it factors through $\partial\colon \Omega P^{2n+2} \longrightarrow U$.*

Proof. Let g be a right homotopy inverse of the H–map ∂ and let e be the homotopy equivalence $e\colon U \times \Omega\Sigma R \xrightarrow{g\cdot\Omega\mathrm{ad}} \Omega P^{2n+2}$ (cf. Theorem 2.1). Define two maps $a, b\colon \Omega P^{2n+2} \longrightarrow \Omega P^{2n+2}$ by the composites

$$
a\colon \Omega\Sigma P^{2n+1} \xrightarrow{e^{-1}} U \times \Omega\Sigma R \xrightarrow{\pi_1} U \xrightarrow{g} \Omega\Sigma P^{2n+1}
$$

$$
b\colon \Omega\Sigma P^{2n+1} \xrightarrow{e^{-1}} U \times \Omega\Sigma R \xrightarrow{\pi_2} \Omega\Sigma R \xrightarrow{\Omega\mathrm{ad}} \Omega\Sigma P^{2n+1}.
$$

In the following diagram

$$\Omega P^{2n+2} \xrightarrow{\Delta} \Omega P^{2n+2} \times \Omega P^{2n+2} \xrightarrow{a \times b} \Omega P^{2n+2} \times \Omega P^{2n+2} \xrightarrow{\mu} \Omega P^{2n+2}$$

$$e^{-1} \searrow \qquad \Big\| $$

$$U \times \Omega \Sigma R \xrightarrow{g \times \Omega \mathrm{ad}} \Omega P^{2n+2} \times \Omega P^{2n+2}$$

the composition along the top row is $a + b$, while the bottom row is the identity map on ΩP^{2n+2}. The commutativity of the diagram gives $\mathrm{Id}_{\Omega P^{2n+2}} \simeq a + b$.

Being an H–map, h is determined by its restrictions on each of the factors of ΩP^{2n+2}, that is, $h \simeq h \circ \mathrm{Id}_{\Omega P^{2n+2}} \simeq h \circ (a + b) \simeq h \circ a + h \circ b$.

If the composite

$$\bigvee_{i=0}^{\infty} P^{4n+3+2mi} \xrightarrow{\Omega \mathrm{ad}} \Omega P^{2n+2} \xrightarrow{h} Z$$

is null homotopic, then we have $h \circ b \simeq *$ and hence $h \simeq h \circ a$ proving the Proposition. As $h \circ \Omega \mathrm{ad}$ is the composite of H–maps, it is itself an H–map. Therefore by the James construction, it is uniquely determined by its restriction to $\bigvee_{i=0}^{\infty} P^{4n+2+2mi}$. The composite

$$\bigvee_{i=0}^{\infty} P^{4n+2+2mi} \xrightarrow{E} \Omega \Sigma \big(\bigvee_{i=0}^{\infty} P^{4n+2+2mi} \big) \xrightarrow{\Omega \mathrm{ad}} \Omega P^{2n+2}$$

is a wedge of mod P Samelson products as it is the adjoint of the wedge of mod P Whitehead products ad: $\Sigma \big(\bigvee_{i=0}^{\infty} P^{4n+2+2mi} \big) \longrightarrow P^{2n+2}$. Being an H–map, h preserves Samelson products. Therefore the wedge of mod P Samelson products $(\Omega \mathrm{ad}) \circ E$, composed with h into the homotopy commutative H–space Z is trivial. $\qquad \square$

Theorem 4.2. *Let Z be a homotopy associative, homotopy commutative H–space. Let $f \colon P^{2n+1} \longrightarrow Z$ be given. Then f extends to an H–map $\overline{f} \colon U \longrightarrow Z$, which is unique up to homotopy.*

Proof. Consider the fibration sequence $\Omega P^{2n+2} \xrightarrow{\partial} U \xrightarrow{*} \bigvee_{i=0}^{\infty} P^{4n+3+2mi}$. Define the map $\overline{f}$ as the composite

$$\overline{f} \colon U \xrightarrow{g} \Omega P^{2n+2} \xrightarrow{\tilde{f}} Z,$$

where $g \colon U \longrightarrow \Omega P^{2n+2}$ is a right homotopy inverse of $\partial \colon \Omega P^{2n+2} \longrightarrow U$ and $\tilde{f} \colon \Omega P^{2n+2} \longrightarrow Z$ is the canonical multiplicative extension of f given by the James construction. Look at the map $\overline{f}$ as a candidate for the multiplicative extension of f. There is a commutative diagram

$$(8) \qquad P^{2n+1} \xrightarrow{E} \Omega \Sigma P^{2n+1} \xrightarrow{\partial} U$$

$$f \searrow \quad \downarrow \tilde{f} \quad \swarrow \overline{f}$$

$$Z$$

where the left triangle commutes by the James construction, while the commutativity of the right triangle is given by Proposition 4.1. Diagram (8) ensures that $\overline{f}$ is an extension of f.

Now we shall prove that $\overline{f}$ is an H–map by showing that the diagram

$$(9) \qquad \begin{array}{ccccccc} U \times U & \xrightarrow{g \times g} & \Omega P^{2n+2} \times \Omega P^{2n+2} & \xrightarrow{\mu} & \Omega P^{2n+2} & \xrightarrow{\partial} & U \\ & \searrow{\overline{f} \times \overline{f}} & \downarrow{\tilde{f} \times \tilde{f}} & & \downarrow{\tilde{f}} & & \downarrow{\overline{f}} \\ & & Z \times Z & \xrightarrow{\mu} & Z & =\!=\!= & Z \end{array}$$

commutes. The left triangle commutes by definition; the middle square commutes since $\tilde{f}$ is an H–map; and the commutativity of the right square is given by Proposition 4.1. Summing this up, diagram (9) commutes.

Finally we are left to show that $\overline{f} = \tilde{f} \circ g \colon U \longrightarrow Z$ is the unique H–map extending $f \colon P^{2n+1} \longrightarrow Z$. To prove this we use the uniqueness of $\tilde{f}$ asserted by the James construction and the result of Theorem 3.1 which establishes that the fibration connecting map $\partial \colon \Omega P^{2n+2} \longrightarrow U$ is an H–map. Let $\overline{f}, \overline{l}$ be two extensions $U \xrightarrow{\overline{f}, \overline{l}} Z$ of the map $f \colon P^{2n+1} \longrightarrow Z$ which are H–maps. Precompose both maps with the H–map $\partial \colon \Omega P^{2n+2} \longrightarrow U$. We obtain two multiplicative extensions $\Omega P^{2n+2} \xrightarrow{\partial} U \xrightarrow{\overline{f}, \overline{l}} Z$ of $f \colon P^{2n+1} \longrightarrow Z$. By the uniqueness of an H–map $\Omega P^{2n+2} \longrightarrow Z$ extending f, it follows that $\overline{f} \circ \partial \simeq \overline{l} \circ \partial$. Precomposing both compositions with the right homotopy inverse $g \colon U \longrightarrow \Omega P^{2n+2}$ of the map ∂, we get

$$\overline{f} \circ \partial \circ g \simeq \overline{l} \circ \partial \circ g.$$

Hence

$$\overline{f} \simeq \overline{l}$$

and the uniqueness assertion is proved. This finishes the proof of the Theorem. $\qquad \square$

The Theorems 3.4 and 4.2 together prove our main result, Theorem 1.1.

5. The exponent for U

5.1. **The H–exponent for $S^{2n+1}\{p^r\}$.** We first reprove Neisendorfer's result (cf. [10]) that $H\exp(S^{2n+1}\{p^r\}) = p^r$ by using the universality of $S^{2n+1}\{p^r\}$.

Theorem 5.1. $H\exp(S^{2n+1}\{p^r\}) = p^r$

Proof. Recall that the degree p^r map on the mod p^r Moore space is null homotopic. Consider the map

$$f \colon P^{2n+1}(p^r) \xrightarrow{p^l} P^{2n+1}(p^r) \xrightarrow{i} S^{2n+1}\{p^r\}$$

given as the composition of the degree p^l map on $P^{2n+1}(p^r)$ with the inclusion of the bottom two cell into $S^{2n+1}\{p^r\}$. As f is a map into a homotopy associative, homotopy commutative H–space, Theorem 1.1 says that it can be extended to a unique H–map $\overline{f}\colon U \longrightarrow S^{2n+1}\{p^r\}$ where U is the universal space of $P^{2n+1}(p^r)$. But $U \simeq S^{2n+1}\{p^r\}$. Any k–power map on $S^{2n+1}\{p^r\}$ is given by the composite $S^{2n+1}\{p^r\} \xrightarrow{\Delta_k} \left(S^{2n+1}\{p^r\}\right)^k \xrightarrow{\mu} S^{2n+1}\{p^r\}$. Since $S^{2n+1}\{p^r\}$ is homotopy commutative, the multiplication μ is an H–map and so it is the k–power map on $S^{2n+1}\{p^r\}$. Taking the p^r–power map on $S^{2n+1}\{p^r\}$ for $\overline{f}$, there is a homotopy commutative diagram

$$
\begin{array}{ccc}
P^{2n+1}(p^r) & \xrightarrow{\ p^l\ } P^{2n+1}(p^r) \xrightarrow{\ i\ } & S^{2n+1}\{p^r\} \\
\downarrow & \nearrow_{\ p^l} & \\
S^{2n+1}\{p^r\}. & &
\end{array}
$$

When $l < r$, the degree map $p^l\colon P^{2n+1}(p^r) \longrightarrow P^{2n+1}(p^r)$ is not null homotopic and therefore neither is its extension $p^l\colon S^{2n+1}\{p^r\} \longrightarrow S^{2n+1}\{p^r\}$. That implies that $H\exp S^{2n+1}\{p^r\} \geq p^r$. On the other hand, for $l = r$, the degree map $p^r\colon P^{2n+1}(p^r) \longrightarrow P^{2n+1}(p^r)$ is null homotopic. Therefore another choice of $\overline{f}$ extending the degree p^r map is the trivial map. As $\overline{f}$ is unique, this implies that $p^r \simeq *$ on $S^{2n+1}\{p^r\}$ and $H\exp(S^{2n+1}\{p^r\}) \leq p^r$. □

5.2. **An upper bound of the exponent for U.** Assume that $n > m$. Consider the attaching map $\Sigma\Theta\colon S^{2n} \longrightarrow S^{2m}$ that defines the two-cell complex P^{2n+1}. It is a suspension of a certain homotopy class of S^{2m-1} and therefore has to be of finite order less than p^m (cf. [2]). Assume that the order of $\Sigma\Theta$ is p^l for some $l \leq m-1$. Then there is a pushout diagram

$$
\begin{array}{ccccc}
S^{2n} & \xrightarrow{\Sigma\Theta} & S^{2m} & \longrightarrow & P^{2n+1} \\
\| & & \downarrow{p^l} & & \downarrow{a} \\
S^{2n} & \xrightarrow{\ *\ } & S^{2m} & \longrightarrow & S^{2n+1} \vee S^{2m}
\end{array}
$$

defining the map $a\colon P^{2n+1} \longrightarrow S^{2n+1} \vee S^{2m}$. We construct a pushout map $\lambda\colon S^{2n+1} \vee S^{2m} \longrightarrow P^{2n+1}$ via the inclusion of the bottom cell $S^{2m} \longrightarrow P^{2n+1}$ and the degree p^l map $p^l\colon P^{2n+1} \longrightarrow P^{2n+1}$ and applying the universal property of pushouts. The resulting map λ then gives the commutative diagram

$$
\begin{array}{ccc}
S^{2n+1} \vee S^{2m} & \xleftarrow{\ a\ } & P^{2n+1} \\
\uparrow & \searrow^{\lambda} & \downarrow{p^l} \\
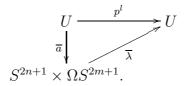S^{2m} & \longrightarrow & P^{2n+1}.
\end{array}
$$

Using the universality of U and the fact that $S^{2n+1} \times \Omega S^{2m+1}$ is a homotopy associative, homotopy commutative H–space, extend the composite $P^{2n+1} \xrightarrow{a} S^{2n+1} \vee S^{2m} \xrightarrow{1 \vee E} S^{2n+1} \times \Omega S^{2m+1}$ to a unique H-map $\overline{a}\colon U \longrightarrow S^{2n+1} \times \Omega S^{2m+1}$.

As $S^{2n+1} \times \Omega S^{2m+1}$ is the universal space of $S^{2n+1} \vee S^{2m}$ (cf. [7]), there is a unique H–extension of the composite $S^{2n+1} \vee S^{2m} \xrightarrow{\lambda} P^{2n+1} \longrightarrow U$ which we denote by $\overline{\lambda}\colon S^{2n+1} \times \Omega S^{2m+1} \longrightarrow U$.

Composing $\overline{a}$ and $\overline{\lambda}$ together, we obtain the following diagram

By the universality of U, the p^l degree map on P^{2n+1} extends to the p^l–power map on U that is an H–map since U is homotopy commutative. Hence $\overline{\lambda}\overline{a} \simeq p^l$ giving the factorisation

Note that $\exp(S^{2n+1} \times \Omega S^{2m+1}) = \max\{\exp(S^{2n+1}), \exp(S^{2m+1})\} = \max\{p^n, p^m\} = p^n$.

Now it follows that the p^{l+n}–power map on U is homotopic to $\overline{\lambda} \circ p^n \circ \overline{a}$ and therefore it is trivial. This proves the following proposition.

Proposition 5.2. *Let p^l be the order of the attaching map $\Sigma\Theta$ defining the two-cell complex P^{2n+1}. Then*

$$H\exp(U) \leq p^{l+n}.$$

5.3. **A lower bound of the H–homotopy exponent for U.** One of the main input data for finding a lower bound of $H\exp(U)$ is the presence of an integer $k \leq m-1$ such that $p^k \Sigma^2 \Theta \simeq *$. That implies the existence of a lift $h\colon S^{2n+1} \longrightarrow U$ of the degree p^k map on S^{2n+1}, that is, a factorisation of the degree p^k map on S^{2n+1} through U

(10)

Knowing that $\exp(S^{2n+1}) = p^n$ (cf. [2]), we can construct a homotopy class of U so that it is not annihilated by p^{n-k-1}. Therefore we prowed the following proposition.

Proposition 5.3. $p^{n-k} \leq \exp(U)$.

Propositions 5.2 and 5.3 together prove Theorem 1.2.

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